Contents/Outline (tentative)

1) Introduction (A Mathematical Stroll through Town)

   a) What?Why?: "A Stroll Through Town" is walking and talking with students about the mathematics in the things they see every day. It is a good way for parents, mentors, and teachers to engage children in learning mathematics.

   b) How?: Seek out interesting mathematics in the everyday world and expand on it. Using the context as a starting point, look in-depth into the computations and concepts.

   c) Purpose?: This is an ongoing project. The concepts discussed in this paper include the following:

      i) multiplication

      ii) similar triangles

      iii) similar triangles and multiplication

      iv) moon mathematics

      v) big numbers

2) Multiplication

   a) Introduction

   b) History

   c) Algorithms

      i) grid

      ii) long and lattice

      iii) Egyptian

3) Similar Tiangles

   a) Introduction

   b) Construction of two types

        i)   parallel lines

        ii)  square root circle

        iii) spiral of Theodorus

    c)  Use in problem solving.

        i)   two ways to estimate the height of a tree

            (1) shadow method

            (2) lumberjack method

4) Similar Triangles and Multiplication

    a)  Finding products on the axes

    b)  Finding fractions, reciprocals, and square roots

5) Moon Mathematics

    a)  Introduction

    b)  Moon/Earth/Sun measurements

    c)  Similar triangles and a solar eclipse

    d)  Bringing down the moon

    e)  Ratios/coincidences

6) Really Big Numbers

    a)  Introduction

    b)  Story of the chess boon

    c)  How much money is there?

    d)  googol and googolplex

7) Summary and Conclusion

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Multiplication

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There are many different ways to think about multiplication as well as many different practical algorithms that can be used to carry out calculations.

To begin our exploration of multiplication, let us start with the multiplication symbol itself. The "X" has been synonymous with multiplication for years, but there are conflicting accounts as to its origin. According to *A History of Mathematics*, it was introduced by William Outred in 1631for religious reasons (Florian Cajori (1919)).
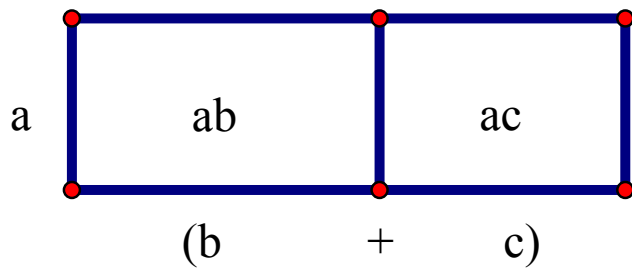
Others claim that the symbol comes from the Egyptians. The Egyptian goddess of weaving, Nieth, is considered to be one of the earliest of the Egyptian goddesses. Her symbol, sometimes found on her crown in depictions, was crossed arrows or knitting needles. The needles represent the warp and weft of knitting, the crossing of two forms of matter to create another. The crossed arrows can be seen as the diagonals of a square. "Upon her four quartered loom, Nieth weaves the four states of 'matter' with which subsequent mother goddesses clothe newborn forms," says mat instructor and author, Michael Schneider (*A Beginner's Guide to Constructing the Universe* (1995) p.72.).

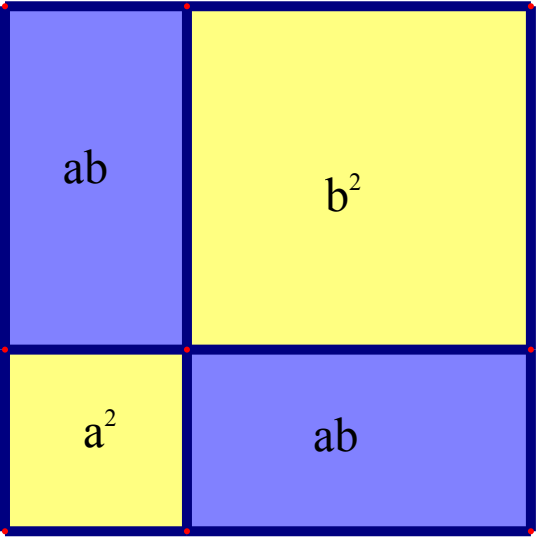| | |
|---|---|
|  |  |
| Nieth with her crossed arrows | Nieth with her knitting needles crown |

Whether we get our symbol from Nieth or from Mr. Outred doesn't really matter in the long run. It is a plausible concept that the "X" comes from the weaving (and wisdom) goddess as multiplication can be thought of as a type of weaving.

Weaving is a good model for thinking about some kinds of multiplication.

Weaving a grid is an expression of the distributive property of multiplication and addition. a (b + c) = ab + ac



What we have are the two areas, ab and ac, created by the weaving of a with b and c.

| | |
|---|---|
|  | Notice here that we are weaving length (a+b) with length (a+b).<br><br>The big square is $(a + b)^2$.<br><br>The yellow regions to the left are also squares, $a^2$ and $b^2$.<br><br>The blue regions are the two rectangles ab.<br><br>So, we can use this weaving technique to see that<br><br>$(a + b)^2 = a^2 + 2ab + b^2$ |

We may want to extend this concept further to see what happens when we square the sum of three or more addends.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ac + 2\,ab + 2bc$$

| c | ac | bc | $c^2$ |
|---|----|----|-------|
| b | ab | $b^2$ | bc |
| a | $a^2$ | ab | ac |
|   | a | b | c |
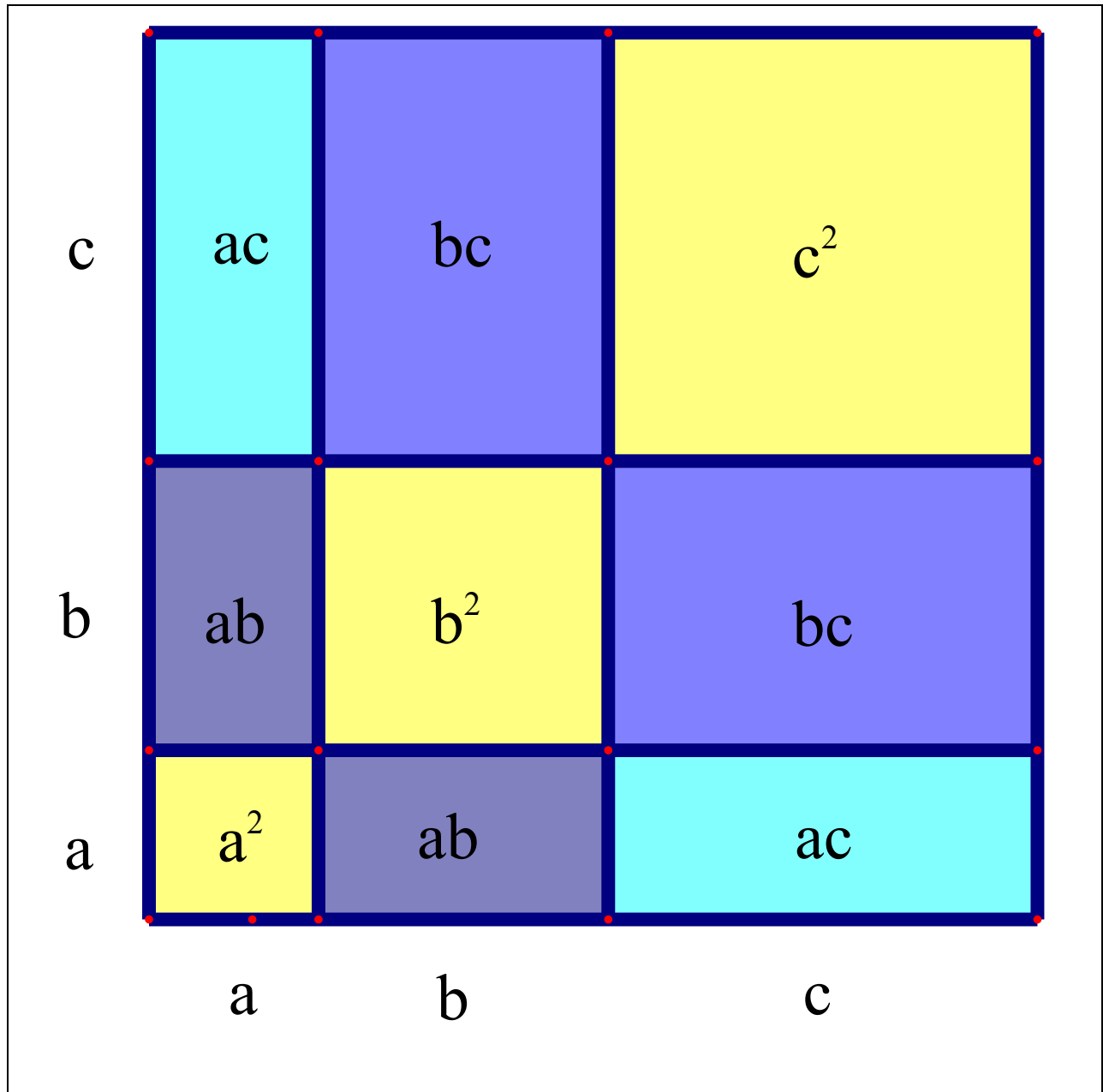
$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$$

| | a | b | c | d |
|---|---|---|---|---|
| d | ad | bd | cd | $d^2$ |
| c | ac | bc | $c^2$ | cd |
| b | ab | $b^2$ | bc | bd |
| a | $a^2$ | ab | ac | ad |

Most multiplication algorithms require that we know the basic multiplication tables, that is, multiplication of the digits.  Here is the table in weaved form.

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | **0** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 0 | 2 | **4** | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 3 | 0 | 3 | 6 | **9** | 12 | 15 | 18 | 21 | 24 | 27 |
| 4 | 0 | 4 | 8 | 12 | **16** | 20 | 24 | 28 | 32 | 36 |
| 5 | 0 | 5 | 10 | 15 | 20 | **25** | 30 | 35 | 40 | 45 |
| 6 | 0 | 6 | 12 | 18 | 24 | 30 | **36** | 42 | 48 | 54 |
| 7 | 0 | 7 | 14 | 21 | 28 | 35 | 42 | **49** | 56 | 63 |
| 8 | 0 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | **64** | 72 |
| 9 | 0 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | **81** |

Notice the perfect squares run the diagonal in the weaving above similar to our previous examples. We could change the orientation of the factors to get a different diagonal of squares. We expect students to memorize this table, to speed up complicated calculations, but, as we have seen, each product is derivable from repeated addition.

Based on my own personal experience, focusing on the squares, in the diagonal center of all of the action, can help orient a child memorizing the "multiplication table" for the first (and only) time. In the process of memorizing the table, students may, like I did, forget a product from time to time. They may have products that just don't stick in their memory. "What is 7 x 6?" For some reason, I would forget. But I know the square 7 x 7 = 49, so, 7 x 6 must equal 7 less, or 42.

******************INSERT Winged Table**********************************

Thinking in terms of squares of two terms can help us think in ways to help find larger and larger squares without a calculator.

We know that $(a + b)^2 = a^2 + 2ab + b^2$
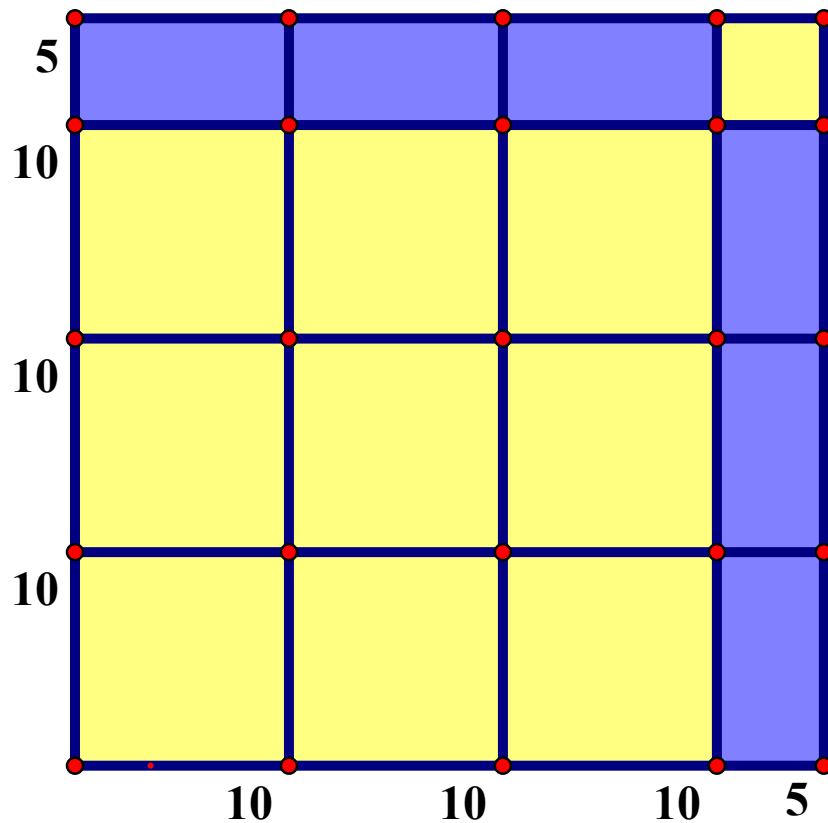


$35^2 = [(3 \times 10) + 5]^2$
$= (3 \times 10)^2 + 2 \times (3 \times 10) \times 5 + 5^2$
$= (3^2 \times 10^2) + (30 \times 10) + 25$
$= 900 + 300 + 25$
$= 1225$

The example above shows how we can extend the notion of weaving a square of two terms to squaring any two digit number.   Similarly, but with many more terms, we can extend the notion of the square of three terms to squaring a three digit number.

As we saw above, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ac + 2\,ab + 2bc$.

So, for example,

$$349^2 = (300 + 40 + 9)^2 = 300^2 + 40^2 + 9^2 + (2\times300\times9) + (2\times300\times40) + (2\times40\times9)$$

$$= 90000 + 1600 + 81 + 5400 + 24000 + 720$$

$$= 121,801$$

Whew!

That is not exactly the easiest way to find the square of a three digit number. The examples above represent an algorithm for multiplication called the grid algorithm.

***************INSERT quadratic factors ********************************

There are two other multiplication algorithms that are popular in schools in the United States. The most popular is known simply as "long multiplication." More rarely used, but just as good an algorithm is "lattice multiplication."

There is not much to say about long multiplication that hasn't been covered well elsewhere. The student should, however, be encouraged to note the similarities between grid multiplication and long multiplication. They are isomorphic after all.

Lattice multiplication is often used as a way for students who lack handwriting skills to demonstrate their multiplication skills sans calculator. The link provided describes the steps well.

There are many algorithms that have been used historically throughout the world to multiply numbers. One type of multiplication algorithm doesn't require knowledge of the multiplication table. All one has to do to use it is to be able to add and to double numbers. This algorithm goes by several names. Most often it is called Egyptian math.

Egyptian math is a way to uses binary numbers, like a computer, to multiply. It often seems like magic to students. The "magic trick" is explained below.

1. Write the two numbers to be multiplied together, x and y, down on paper;

2. Under one number, x, make a column (c 1) of numbers, starting with 1 and doubling each row until you have gone past the number at the top.

3. Under the next number, y, write the number, then double it in each row of the column (c 2) for as many rows as listed in step 2.

4. Now decide which numbers from c 1 add to give the number, x, and mark those rows.

5. Mark the corresponding rows in c 2.  Add the marked numbers for the product of x and y.

Let's look at 35 x 35 done this way:

| <u>35</u> | Part of 35? (1=yes, 0=no) | <u>35</u> | Total |
|-----------|---------------------------|-----------|-------|
| 1 | 1 | 35 | 35 |
| 2 | 1 | 70 | 70 |
| 4 | 0 | 140 | 0 |
| 8 | 0 | 280 | 0 |
| 16 | 0 | 560 | 0 |
| 32 | 1 | 1120 | 1120 |
| 64 | 0 | 2240 | <u>0</u> |
| Product: | | | 1225 |

Teachers: Do a couple of these on the board before you explain the mathematic behind the mystery.  It will seem like magic if done with skill.

The algorithm is based on writing one of the factors as a binary number and multiplying by corresponding doubling products of the other factor.

$35 = 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 + 0 \times 2^4 + 1 \times 2^5$

If we distribute 35 to each of the terms in the binary expansion, we get

$35 \times 35 = 35 \times (1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 + 0 \times 2^4 + 1 \times 2^5)$

$$= 35 \times 2^0 + 35 \times 2^1 + 0 \times 2^2 + 0 \times 2^3 + 0 \times 2^4 + 35 \times 2^5$$

Each row of the column of the second number represents $35 \times 2^n$. So the nonzero digits show us which doublings to add.
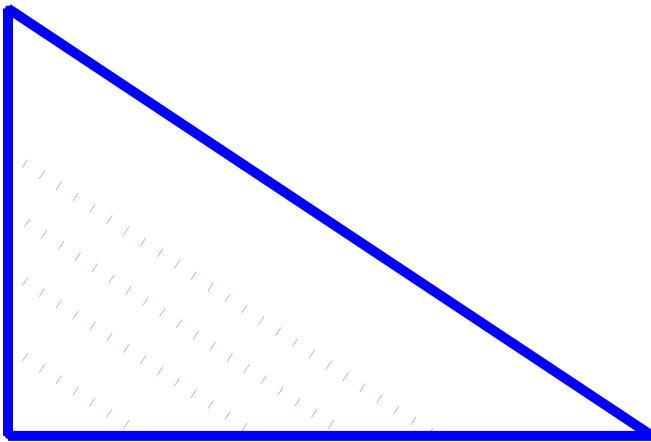
Similar Triangles

Triangles are similar if they are proportionate copies of each other. They must have the **same shape,** but they can be **any size**. This means that all angles are the same.

Similar triangles can be useful in solving many different types of problems including the tree height problem, as we will see below.

First, a word on a couple of ways that similar triangles can be formed: The most common way to form similar triangles is by cutting a triangle with a line parallel to one of the sides of the triangle.

In the triangle below, four segments have been inscribed inside the triangle, each parallel to a side. This creates 5 similar triangles.



Of course, this type of construction will work for segments parallel to any side of any type of triangle. Just to get a picture, let's look at another example. In the picture below, lines that don't touch are parallel. How many similar triangles are shown?
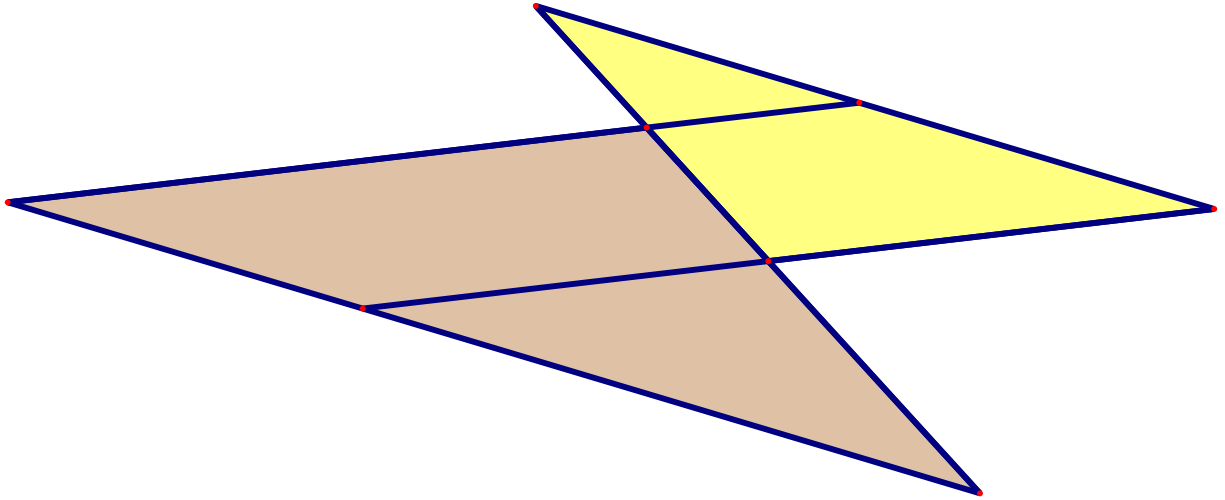
Another type of similar triangle construction is useful in the construction of square roots. In this construction, no parallel lines are drawn.

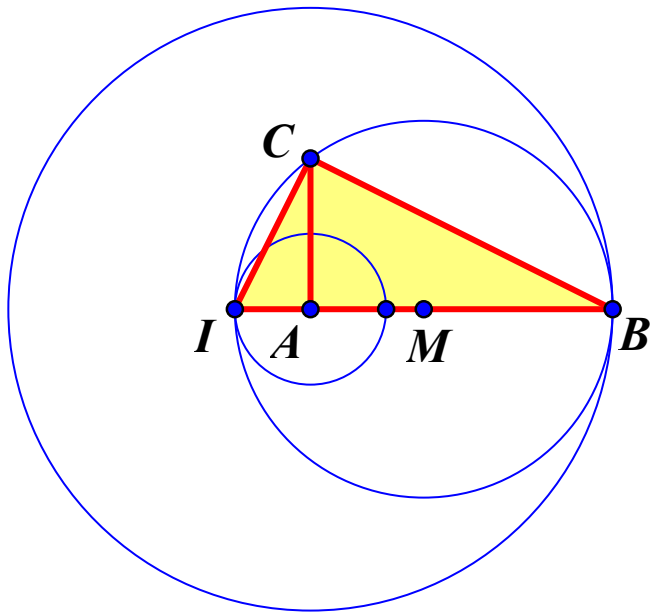| Step by Step: | |
| --- | --- |
| Start with distance $|AB|$ (This is the number to find the root of .)<br><br>Extend AB one unit to I. $|IA|=1$.<br><br>Find the midpoint, M, of IB.<br><br>Construct a circle centered at M, with radius $|MB|$.<br><br>Construct a perpendicular to IB through A.<br><br>Let C be the intersection of the circle and the perpendicular line.<br><br>Claim: $|AB|^2 = |AC|$ |  |

To show that AC is the square root of AB, we need to understand that the two triangles, $\Delta$ABC and $\Delta$ACI, are similar.

We can see this by rotating the triangle ACI about the point A. Both angles at A are right. I rotates up to I'. C rotates down to C'. How can we tell if the angles are the same?

Another property of similar triangles, in addition to having equal angles, is the ratios of the sides are proportional. In the example above, we know that

$|IA|/|AC| = |AC|/|AB|$, and $|IA| = 1$. Cross multiplying, we get

$|AB| = |AC|^2$

Back to the tree problem: How can we use similar triangles to tell the height of the tree (or anything else within a reasonable range)? There are at least two ways.

One method uses the sun's rays and shadows. Using the assumption that the sun's rays are essentially parallel, we can construct two right triangles with parallel hypotenuses if we have an object (or person) of known height.
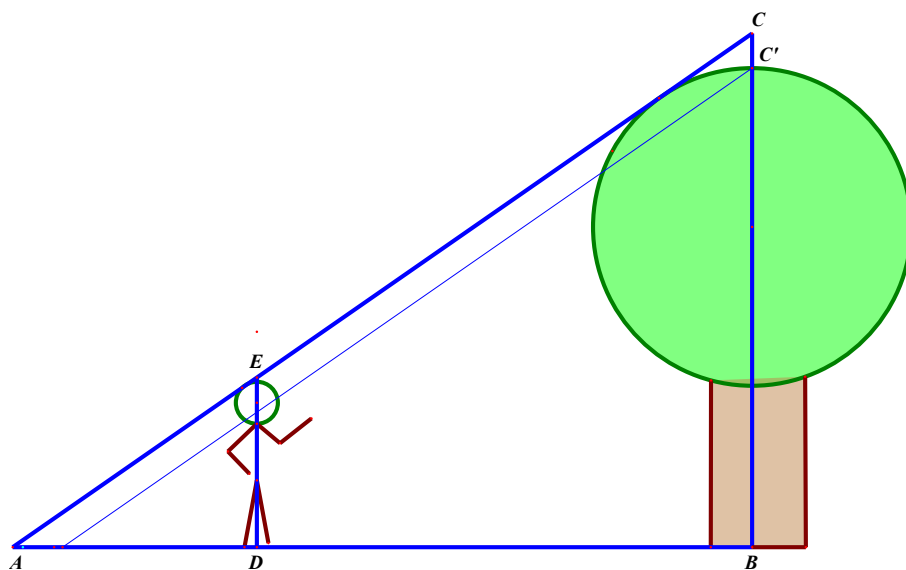


We start with the big blue triangle made by the sun and the tree. The length $|AB|$ is the length of the shadow of the tree to the center of the trunk. We measure $|AB|$ to

be 38 feet.  The other two lengths of the triangle are unknown.  Now we need a person or object of known height.  Say Ed is 5'6" tall, that is, $|ED| = 5.5$ feet.  His shadow is 5.2 feet long.  So $|AD| = 5.2$ feet.  We now have all we need to solve the problem.  We use our three known lengths to find the height.  Specifically,

$|BC|/|ED| = |AB|/|AD|$

$|BC| = (|AB||ED|)/|AD| = (38 \times 5.5)/5.2 = 40.1923\ldots$

The height of our tree using this method is estimated to be about 40 feet.

Notice the thin blue line in the picture above.  There may be some error involved due to the shape of the tree.

There is a better way to do the same problem that lumberjacks have used for years.  It is important when cutting down a tree to know how tall it is so you know how far it will fall - and how loud you will have to yell, "Timber."  Time is also an important consideration on the job, so a more efficient way to tell the height of trees would be desirable.

So you are a lumberjack.  Here is the lumberjack method for estimating the height of a tree.  First, measure off a piece of wood the size of your arm.  An axe handle will do.  Now hold your arm parallel to the ground and the axe handle parallel to the tree. (On level ground, you will hold the axe handle at a right angle to your arm.) Sight the top of the tree over the top of the axe handle.  Where you are standing plus your height is about where the top of the tree will fall.  Measure from where you are to the center of the tree trunk, and then add your height.  Now you have a good estimate of the tree's height.

***********************ADD PICTURE************************************


The tie-in to multiplication:

In the Cartesian plane, similar triangles can be used as a model for multiplication.  If the side of one triangle is a unit, then there are proportions that can be used to determine products and fractions.

**************************ADD PAPER – TMM****************************

*****************INSERT MOON MATH**************************************

**Big Numbers**

Kids love big numbers.

I was very impressed one day when we were on one of our walks, my son, who was five years old at the time, asked me a profound question about numbers:

"Is there a biggest number?"

I told him that there wasn't a biggest number. "Think of the biggest number you can think of," I said. "Now add 1. The number that you have now is a bigger number than the number that you started with. So, no matter how big a number you can think of, you can always get a bigger number by adding one."

"What if I just can't think of the biggest number?" he said.

I had to think about that for a while. What if, in some context, there is a number that is the biggest number, beyond which; even the addition of one is meaningless. That concept is defeated through abstraction, but in concrete terms, in a finite universe, could there be a largest number?

*Once upon a time:*

*Long ago, there was a powerful king who was given a gift of the game of chess. This king was so impressed*

with the game that it pleased his majesty to offer a royal boon (that is, a really big royal favor) as a way of thanking the person who gave it to him.

As was the tradition, the king had the man who gave him the gift brought to his throne room with much fanfare. Trumpets blared as the man was lead to the front of the hall. He was introduced by the court crier, "The man who gave you the chess set, your majesty."

The man, whose name happened to be Gerald, bowed so low his gray hair swept the floor.

"This game you have given us pleases us very much!" said the king.

"I am glad that your highness finds my humble gift of the chess board, its 64 squares and its 32 pieces, a suitable amusement, your majesty," said the man, who happened to be a math teacher from a nearby village.

"We certainly do appreciate thy gift, and we wish to thank thee!  Now,…" the king paused for the trumpets to play a few short blasts, and then, "…ask what you will, my kingdom to fulfill," said the king with much enthusiasm and pride.

"Well, your majesty, things are kind of tight at my high school, and well, some money would be nice." The man proceeded as the king nodded. "I would simply like a single copper (about 100th of a rupee) placed on a corner square of the chess board, next to that, in the next square, place two coppers. Continue to double the amount until the board is filled. Quite simple. That should be plenty of money for me and my school. That is all, your highness."

Without consulting his vizier, the king agreed and ordered a jar of coppers brought up from the royal vault.

That was soon gone, as servants stacked the coins in the prescribed manner. Another jar, and then another was ordered brought up. Next a cart was sent. The board was filling up as coins were being stacked to the ceiling of the throne room! A train of camels was needed to bring more coppers.

Finally, the large stacks of coins refused to stand and came crashing down, filling the floor of the kings throne room.

The king, who was running out of coppers, decided that he had to make an executive decision and call the boon

fulfilled on the technicality of the request being too big for the chess board.  Gerald was okay with that as he had plenty of cash for himself and his high school by this time.

They all lived happily ever after, but the king always brought his calculator to boon granting ceremonies from then on.

The End.

Question:

How much money, in coppers, would Gerald have if the king had been able to fill the whole chess board?

Answer:

The amount of money needed to fill the chess board by doubling is,

$$\sum_{i=1}^{64} 2^i$$

$1 + 2 + 4 + \cdots + 2^{64}$ or

Now, $2^{64}$ = 18,446,744,079,551,616 coppers!  That is, 18.45 quadrillion coppers or 18.45 trillion rupees!  That alone is much more than most royal treasuries. The sum is very large indeed.

$$\sum_{i=1}^{64} 2^i = 36,893,488,147,419,103,231$$

Now that is a big number, and, in context of dollars, or rupees, that is about 36.8 quadrillion rupees.  If a rupee is roughly the same as a dollar, that is more money than is!  (See below.) That is about as big as it gets.
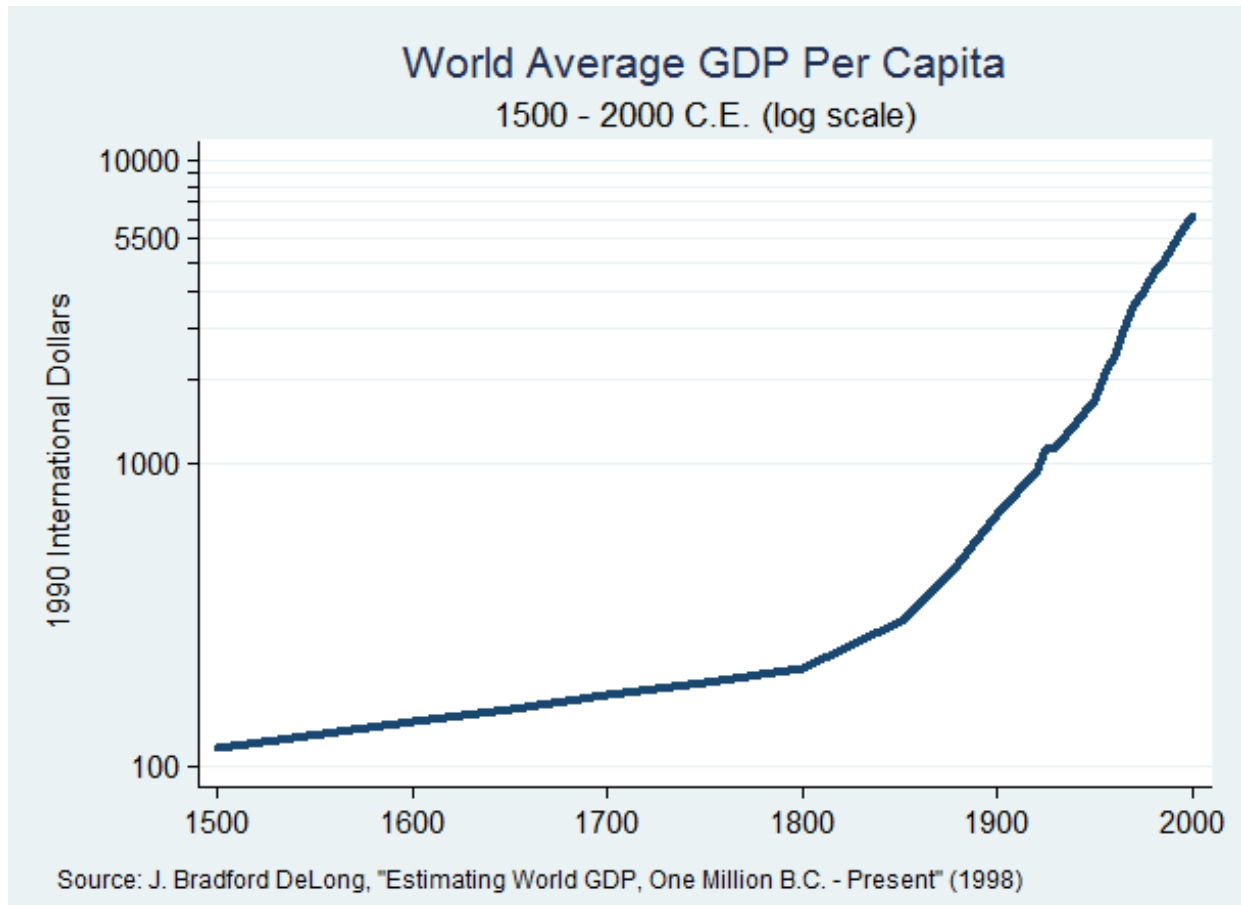
Question:

How much money, in terms of US dollars, is there in the world?

Answer:

No one really has a good answer for this, and there are various ways to estimate it. Do you include the estimated value of all of the natural resources on Earth?  What about the collateral loss of value associated with overuse of a resource?  The value of this as a math problem is in the various efforts students might employ to justify an answer.  A suggested method is described below.

An answer can be given in context of world GDP.  Instead of just looking up the current world GDP numbers, make a math problem out of it.  Take the estimated world GDP per capita data and model a projection to the current date.  Next, multiply by the current estimated world population.

Kip Tanner



## World Average GDP Per Capita
### 1500 - 2000 C.E. (log scale)

Source: J. Bradford DeLong, "Estimating World GDP, One Million B.C. - Present" (1998)

As can be seen above, about the time the industrial revolution kicks in, the mid 1800s, the graph gets rather linear. An argument can be made for a best fit curve, but, considering the world economic downturn, a line seems more appropriate to project estimates. Take the following three points and construct a best fit line: (1850,500) , (1900,1000), and (2000,5500).
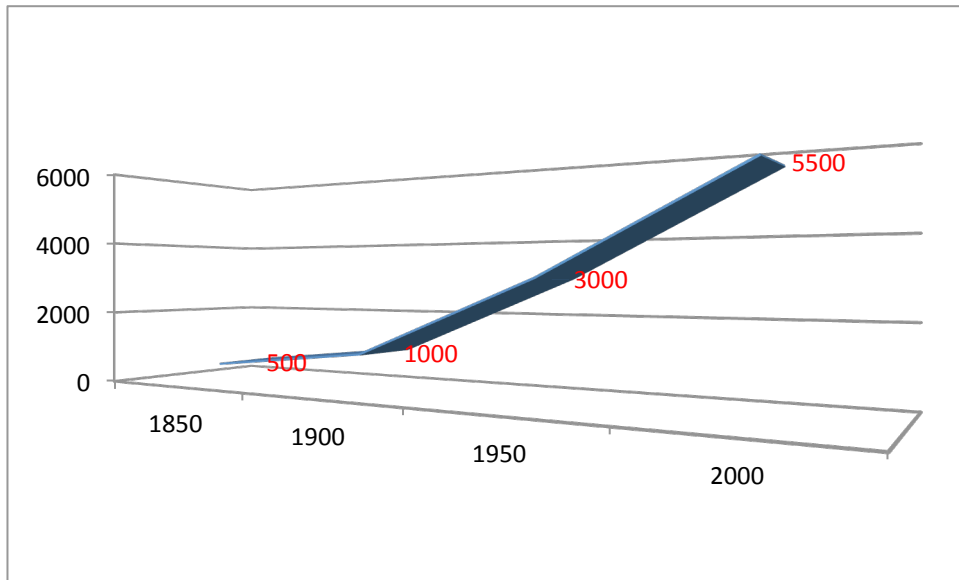
We want to find a best fit line by the least square method.  (A small data set like this is a good way to introduce this concept.)

1.  Calculate the arithmetic mean of each set of coordinates.

2.  Compute the sum of the squares of the range of dates (x axis).

3.  Compute the sum of each date (x value) multiplied by its GDP (y value).

4.  Calculate the slope

$$m = \frac{\sum xy - \frac{(\sum x)(\sum y)}{n}}{\sum x^2 - \frac{(\sum x)^2}{n}}$$

$n$ = number of data points.

5.  Calculate the y intercept as the arithmetic mean of the y values minus the product of the slope and the arithmetic mean of the x values.

The resulting equation is:

y = 34 x - 62950

When x is 2013 (the date of this paper), the estimated world GDP by this model is 5492. We would expect this to be close to 5500, but more.  Something must be wrong with our model.  If we insist on a linear model, we are going to have to throw out the first data point. If we do this we get:

y = 45 x – 84583

Now we get about 6000 for the current year.

Now, we multiply this result by the current estimated world population, and get

Estimated world GDP = (6,000)(7,100,000,000) = 42,600,000,000,000 or 42.6 trillion.

This is about half of the published GWP (gross world product) which uses different methods to calculate per capita estimates.

Any way you look at it, **100 trillion is about as big as it gets right now, in context of all the money in the world.**

But when we start talking in context of astronomy and physics, the numbers can get very big indeed.  Here are some names of base ten periods we might need to know to talk about really big numbers.

1) units           $= 10^0$
2) thousands     $= 10^3$
3) millions       $= 10^6$
4) billions        $= 10^9$
5) trillions       $= 10^{12}$
6) quadrillions   $= 10^{15}$
7) quintillions    $= 10^{18}$
8) sextillions     $= 10^{21}$
9) septillions     $= 10^{24}$
10) octillions     $= 10^{27}$
11) nonillions    $= 10^{30}$
12) decillions     $= 10^{33}$
13) undecillions   $= 10^{36}$
14) duodecillions   $= 10^{39}$
15) tridecillions    $= 10^{42}$

Perhaps when you look at the word names and the magnitude of each of the periods, you notice something isn't quite right.  So you are saying to yourself, "I know Latin prefixes when I see them, and those don't match the powers at all."

That is correct, because the million is the starting point for the naming system. Even though the prefix mil- suggest 1000, we all know that 1,000,000 = 1,000 x 1,000.  The subsequent prefixes are numeric: bi- means 2, tri- 3, quad- 4, and so on.  A function

that can be used to get from the power of ten to the Latin prefix that matches the period is given below:

Let n be a power of 10.  f(n)= (n/3) – 1.  This function only works if I start with one of the powers listed above.  A better function would take into account the powers in between.  Here, modular math works fine.

****************************INSERT conversion*********************************

Numbers can get very big indeed.  After all, when we are talking about the universe, we are talking about everything there is, right?

One famous large number is the googol.  (The ubiquitous internet company, Google, owes its name to a misspelling of this number!)  This number is 10 to the 100$^{th}$ power, that is, 1 with 100 zeroes.  By our Latin word converter function, we have

*************************************INSERT***********************************

****************************Conclusion*************************************